在高考純數科，設計一道好的考題(特別是長題目)並不容易，很多時那些環環緊扣的長題目其實本身有著一定的數學意義。對出題者而言，要憑空想出這些數學結果是不太可能的(實際上這是數學研究者的工作)，於是他們便常常「就地取材」，尋找一些已被研究出來的數學結果，改頭換面成爲一道純數題目。對考生而言，如果你能對這些常用的數學題材有所認識，很可能會對你應付這類題目有所幫助。

今次要談的題材是 Legendre polynomial 和 Rodrigues formula，在 91 年(paper II Q13)曾經完整地出現，而在近年(05 paper II Q10 及 10 paper II Q8)亦有其踪影，值得同學參考。

我們先來看看 91 年卷 II 第 13 題的設計：

For $n=0,1,2,\dots$, let $R_n(x)=(x^2-1)^n$ and $P_n(x)=R_n^{(n)}(x)$. 定義了何謂 $P_n(x)$，題目便要求考生利用這定義，證明一大堆關於 $P_n(x)$ 的性質。

在數學而言，這裏的 $P_n(x)$ 實際上就是勒讓德多項式(Legendre polynomials)，算是一類非常有用的特殊函數，在物理的電磁學計算中非常有用，在維基百科中鍵入 Legendre polynomials 亦可找到。與 $P_n(x)$ 相類似的還有其他特殊多項式，10 年 paper II Q8 也是使用類似的題材來設計題目。

$P_n(x)$ 可以用不同的方法去定義，例如：

定義 1：利用微分方程：$P_n(x)$ 定義爲

$(1-x^2)\frac{d^2}{dx^2}P_n(x)-2x\frac{d}{dx}P_n(x)+n(n+1)P_n(x)$ 的解。

定義 2：利用生成函數(generating function)，$P_n(x)$ 定義爲 $\frac{1}{\sqrt{1-2xt+t^2}}$ 對 t 展開式中 t^n 項的係數，$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n$。

定義 3：利用羅德里格公式(Rodrigues formula)，直接定義

$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$。

三個定義互相等價，由其中一個出發可導出另外兩個。但由於 AL 純數沒有提及生成函數，所以定義 2 從沒在試卷中出現。91 年的題目就是利用了定義 3，只是把 $\frac{1}{2^n n!}$ 消掉。我們就試試採用定義 3，

AL Pure maths notes: Legendre Polynomials

$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right]$，推導下列關於 $P_n(x)$ 的性質。同學試試跟著演算一次，並留意當中的關鍵步驟。

性質 1：$P_n(x)$ 是一個 x 的 n 次多項式。(91II13 (a)(i))

性質 2：任何 n 次多項式 $f(x)$ 可用 $P_n(x)$ 展開，即 there exists $c_k \in \mathbb{R}$ such that $f(x)=\sum_{k=0}^{n} c_k P_k(x)$ for all polynomial $f(x)$ of degree n，變相指 $P_n(x)$ 是完備的(complete)。(91II13 (a)(ii))

性質 3：$P_n(x)$ satisfies $\left(1-x^2\right)\frac{d^2}{dx^2}P_n(x)-2x\frac{d}{dx}P_n(x)+n(n+1)P_n(x)$，即定義 1。(91II13 (b))

性質 4：正交性(orthogonality)，當 $m \neq n$ 時，$\int_{-1}^{1} P_m(x)P_n(x)\,dx=0$；只有在 $m=n$ 時，$\int_{-1}^{1} P_m(x)P_n(x)\,dx=\frac{2}{2n+1}$。這亦與性質 2 相關，正交性給出了計算 c_k 的方法，$c_k=\frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)\,dx$。(91II13 (c))

性質 5：與性質 4 相關，計算 $\int_{-1}^{1} f(x)P_k(x)\,dx$：for any polynomial $f(x)$，

$\int_{-1}^{1} f(x)P_k(x)\,dx=\frac{(-1)^k}{2^k k!}\int_{-1}^{1}\left(x^2-1\right)^k\frac{d^k f(x)}{dx^k}\,dx$。(05II10 (c))

以下爲證明部分，爲便於同學理解，我用英文來寫。

Definition: $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right]$

Property 1: $P_n(x)$ is a polynomial of order n in x.

Proof:

Since $\left(x^2-1\right)^n=x^{2n}+\text{lower order terms}$,

$\frac{d^n}{dx^n}\left(x^2-1\right)^n=\frac{(2n)!}{n!}x^n+\text{lower order terms}$ which is a polynomial in x of order n.

Property 2: There exists $c_k\in\mathbb{R}$ such that $f(x)=\sum_{k=0}^{n}c_kP_k(x)$ for all polynomial $f(x)$ of degree n.

Proof: This is done by second principle of mathematical induction on n. (you need to assume the statement is true for all $0,1,2,\ldots,j$ to prove $T(j+1)$)

Let $T(n)$ be the statement.

For $T(0)$, $f(x)=c$ for some constant c (since $f(x)$ is a polynomial of zeroth order). $P_0(x)=1$ obviously. Hence $f(x)=cP_0(x)$ and $T(0)$ is true.

Assume the statement is true for $0,1,2,\ldots,j$ for some non-negative integer j.

For, $T(j+1)$, consider $f(x)$ is a polynomial of degree $j+1$.

Since $P_{j+1}(x)$ is a polynomial of degree $j+1$ and there is no x^{j+1} term in $P_0(x),P_1(x)\ldots$up to $P_j(x)$, we can choose c_{j+1} such that $f(x)-c_{j+1}P_{j+1}(x)$ does not contain x^{j+1} term.

Hence $f(x)-c_{j+1}P_{j+1}(x)$ is a polynomial of degree less than $j+1$ (may not be exactly j), and by assumption, $f(x)-c_{j+1}P_{j+1}(x)=\sum_{k=0}^{j}c_kP_k(x)$.

Therefore $f(x)=\sum_{k=0}^{j+1}c_kP_k(x)$ and $T(j+1)$ is also true.

Property 3: $P_n(x)$ satisfies $\left(1-x^2\right)\frac{d^2}{dx^2}P_n(x)-2x\frac{d}{dx}P_n(x)+n(n+1)P_n(x)$.

Proof: Start from consider $\left(x^2-1\right)^n$ and differentiate $n+2$ times.

Consider $f(x)=\left(x^2-1\right)^n$.

Differentiate with respect to x, $f'(x)=2nx\left(x^2-1\right)^{n-1}$.

Multiply both sides by $\left(x^2-1\right)$ to get $\left(x^2-1\right)f'(x)=2nxf(x)$.

Differentiate once more to obtain $\left(1-x^2\right)f''(x)+2(n-1)xf'(x)+2nf(x)=0$.

Differentiate n times using Leibniz theorem, we get

$\left(1-x^2\right)\frac{d^2}{dx^2}P_n(x)-2x\frac{d}{dx}P_n(x)+n(n+1)P_n(x)$.

Property 4: when $m\neq n$, $\int_{-1}^{1}P_m(x)P_n(x)dx=0$; when $m=n$ $\int_{-1}^{1}P_m(x)P_n(x)dx=\frac{2}{2n+1}$

Proof:

We first proof the case for $m\neq n$.

First cast the relation in property 3 into this form

$-\frac{d}{dx}\left[\left(1-x^2\right)\frac{d}{dx}P_n(x)\right]=n(n+1)P_n(x)$.

Multiply by $P_m(x)$ and integrate,

$$\begin{aligned}
n(n+1)\int_{-1}^{1}P_m(x)P_n(x)dx &= -\int_{-1}^{1}P_m(x)\frac{d}{dx}\left[\left(1-x^2\right)\frac{d}{dx}P_n(x)\right]dx \\
&= -\left[\left(1-x^2\right)P_m(x)\frac{d}{dx}P_n(x)\right]_{-1}^{1}+\int_{-1}^{1}\left(1-x^2\right)P_m'(x)P_n'(x)dx \\
&= \int_{-1}^{1}\left(1-x^2\right)P_m'(x)P_n'(x)dx
\end{aligned}$$

Note that the right hand side is symmetric in m,n so that

$$n(n+1)\int_{-1}^{1}P_m(x)P_n(x)dx=m(m+1)\int_{-1}^{1}P_m(x)P_n(x)dx$$

If $n\neq m$, then the integral $\int_{-1}^{1}P_m(x)P_n(x)dx$ must be zero.

The case of $m=n$ is tedious, the basic strategy is to do integration by part n times and reaching $\int_{-1}^{1}\left(1-x^2\right)^n dx$, then evaluate the integral. The calculation is lengthy and hence skipped.

Property 5: for any polynomial $f(x)$, $\int_{-1}^{1} f(x)P_k(x)\,dx = \frac{(-1)^k}{2^k k!}\int_{-1}^{1}(x^2-1)^k \frac{d^k f(x)}{dx^k}\,dx$.

Proof: This requires an important result: $\left.\frac{d^m}{dx^m}(x^2-1)^n\right|_{x=1} = \left.\frac{d^m}{dx^m}(x^2-1)^n\right|_{x=-1} = 0$

for $m<n$. (which is 05II Q10(b)).

This can be proved by considering $(x^2-1)^n = (x+1)^n (x-1)^n$, which upon differentiating m times ($m<n$) must remain factors of $(x+1)(x-1)$ in each term. Hence the value evaluated at $x=-1$ or $x=1$ must be zero.

To prove property 5,

$$\int_{-1}^{1} f(x)P_k(x)\,dx = \frac{1}{2^k k!}\int_{-1}^{1} f(x)\frac{d^k}{dx^k}(x^2-1)^k\,dx$$

Note $\frac{d^k}{dx^k}(x^2-1)^k\,dx = d\left[\frac{d^{k-1}}{dx^{k-1}}(x^2-1)^k\right]$, hence we can integrate by parts to get

$$\begin{aligned}\int_{-1}^{1} f(x)P_k(x)\,dx &= \frac{1}{2^k k!}\left[f(x)\frac{d^{k-1}}{dx^{k-1}}(x^2-1)^k\right]_{-1}^{1} - \frac{1}{2^k k!}\int_{-1}^{1} f'(x)\frac{d^{k-1}}{dx^{k-1}}(x^2-1)^k\,dx \\ &= -\frac{1}{2^k k!}\int_{-1}^{1} f'(x)\frac{d^{k-1}}{dx^{k-1}}(x^2-1)^k\,dx \\ &= \cdots \\ &= \frac{(-1)^k}{2^k k!}\int_{-1}^{1}(x^2-1)^k \frac{d^k f(x)}{dx^k}\,dx\end{aligned}$$

Basically we do integration by parts k times to transfer all the $\frac{d}{dx}$ from $(x^2-1)^k$ to $f(x)$.

Text: Ko Tze